AN ENSEMBLE AVERAGING APPROACH TO OPTIMAL GUIDANCE POLYNOMIAL APPROXIMATIONS

Henry J. Kelley

and

Walter F. Denham

Contract NAS 8-11241

November 1964

Analytical Mechanics Associates, Inc.
Uniondale, L.I., N.Y.

AN ENSEMBLE AVERAGING APPROACH TO OPTIMAL GUIDANCE POLYNOMIAL APPROXIMATIONS*

Henry J. Kelley** and Walter F. Denham***
Analytical Mechanics Associates, Inc.

ABSTRACT

14117 8

An approach to optimal guidance synthesis is developed in which an ensemble-averaged second order approximation to the performance function is minimized subject to constraints on the means and variances of other functions. The minimization is with respect to coefficients of assumed polynomial approximations of a linear feedback control law (in which the state is perfectly known) and coefficients in a linear termination law. A brief comparison is drawn with deterministic neighboring extremal control. While attention is directed mainly to first order necessary conditions, some comments are made on numerical solution by first and second order successive approximation methods. Extensions to include disturbances other than initial errors and to include state estimation errors are discussed briefly.

Author

---

* This research was performed under Contract NAS 8-11241 with the Aero and Astrodynamics Division of NASA Marshall Space Flight Center, Huntsville, Alabama.

** Vice-President

*** Senior Analyst

Introduction

The earliest theoretical approaches to optimal guidance (Refs. 1 and 2) lead to computational methods for synthesizing linear feedback systems furnishing an approximation optimal to second order in an expansion about a given optimal reference trajectory. While the resulting systems fulfill their theoretical promise in providing high performance, terminal accuracy is found to be wanting, and the practical mechanization of the feedback law is encumbered by the need for storing time-varying "gains". Recent studies of the terminal accuracy problem (Refs. 3 and 4) indicate that a large improvement may be realized by transverse state comparison with the reference trajectory and suggest that this relatively simple procedure may be more effective than the addition of quadratic terms in the feedback approximation.

The present paper reports an idea for a synthesis scheme in which an ensemble-averaged second order approximation to the performance index is minimized with respect to certain parameters. These parameters include the coefficients in three polynomials in time which are used in place of general time-varying functions. Polynomial approximations are used for (1) the control programs of the optimal reference trajectory; (2) the state variable histories of the optimal reference trajectory; (3) the feedback gains for the assumed linear feedback control system. Additional parameters to be optimized are the coefficients in an assumed linear rule for termination of perturbed trajectories. The treatment is based upon the statistical methods pioneered in Refs. 5 and 6 in connection with synthesis of optimal midcourse guidance approximations.

Formulation of the Problem

The dynamical system under consideration satisfies

$$\dot{x} = f(x, u, t) \tag{1}$$

where

$x(t)$ is an n-vector of state variables

$u(t)$ is an m-vector of control variables

t is the independent variable (hereafter called time)

f is an n-vector of known functions of x, u, t

$(\dot{\ })$ is $\frac{d}{dt}(\ )$

The system operates over a finite time interval. The initial time t_o is assumed fixed, but the initial state is a vector of random variables with specified ensemble average properties. The problem is to minimize the ensemble average of a given function of the terminal conditions*

$$J = \mathcal{E}\{\varphi[x(t_f), t_f]\} \tag{2}$$

subject to the constraints

$$\mathcal{E}\{\psi[x(t_f), t_f]\} = 0 \tag{3}$$

$$\mathcal{E}\{(\psi^j[x(t_f), t_f])^2\} = N^j \tag{4}$$

where g^j is the j^{th} component of any vector g. J is to be minimized while specifying the means and variances of the functions ψ^j.

---

* t_f is the terminal time

It is assumed that nominal* control programs $\bar{u}(t)$ have been determined which minimize $\varphi[\bar{x}(\bar{t}_f), \bar{t}_f]$ while meeting constraints $\psi[\bar{x}(\bar{t}_f), \bar{t}_f] = 0$. Thus,

$$\dot{\bar{x}} = f(\bar{x}, \bar{u}, t) \tag{5}$$

$$\dot{\lambda} = - \left(\frac{\partial f}{\partial x}\right)^T \lambda \tag{6}$$

$$\lambda^T \frac{\partial f}{\partial u} = 0 \qquad \left(\frac{\partial^2 f}{\partial u^2} \neq 0 \text{ is assumed}\right) \tag{7}$$

with boundary conditions

$$t_o, \ \bar{x}(t_o) \text{ specified} \tag{8}$$

$$\psi[\bar{x}(\bar{t}_f), \bar{t}_f] = 0 \tag{9}$$

$$\lambda^T(\bar{t}_f) = \left(\frac{\partial \varphi}{\partial x} + \bar{\nu}^T \frac{\partial \psi}{\partial x}\right)_{t=\bar{t}_f} \tag{10}$$

$$(\lambda^T f)_{t=\bar{t}_f} = - \left(\frac{\partial \varphi}{\partial t} + \bar{\nu}^T \frac{\partial \psi}{\partial t}\right)_{t=\bar{t}_f} \tag{11}$$

where $(\ )^T$ is the transpose of $(\ )$, the ij^{th} element of a matrix $\frac{\partial g}{\partial y}$, g and y both vectors, is $\frac{\partial g^i}{\partial y^j}$. With $\bar{u}(t)$ and $\bar{x}(t)$ specified, the analysis will be carried out in terms of the perturbation quantities $\delta u(t)$ and $\delta x(t)$, where, by definition

$$u(t) = \bar{u}(t) + \delta u(t) \tag{12}$$

$$x(t) = \bar{x}(t) + \delta x(t) \tag{13}$$

---

* $(\bar{\ })$ is $(\ )$ evaluated on the nominal path.

The minimization of J is to be carried out with respect to a number of parameters of the problem. One set of these parameters appears in the rule for terminating trajectories which must be imposed because there is no automatic way to determine t_f on each member of the ensemble. Suppose that the termination rule is described by

$$\Omega[x(t), t]_{t=t_f} = 0 \tag{14}$$

where Ω may be any once differentiable function of x and t. Consistent with the second order approximation theory to be employed, the optimality of the reference trajectory leads to the result that the most general Ω relevant in the analysis is a linear function of x and t. To first order, then, (14) may be written as

$$0 = \Omega[\bar{x}(\bar{t}_f), \bar{t}_f] + [\frac{\partial\Omega}{\partial x}\delta x]_{t=\bar{t}_f} + \dot{\Omega}\, dt_f \tag{15}$$

where, by definition,

$$\dot{\Omega} = \left(\frac{\partial\Omega}{\partial x}\dot{x} + \frac{\partial\Omega}{\partial t}\right)_{t=\bar{t}_f}$$

Since $t_f = \bar{t}_f + dt_f$, the terminal time may be determined from (15) provided $\dot{\Omega} \neq 0$. This is simply the statement that Ω must not be a constant of the motion if (15) is to give a solution for t_f.

Solving (15) for dt_f gives

$$dt_f = \bar{\Omega} + \Omega_x\, \delta x(\bar{t}_f) \tag{16}$$

where, by definition,

$$\bar{\Omega} = \frac{\Omega[\bar{x}(\bar{t}_f), \bar{t}_f]}{(-\dot{\Omega})}, \qquad \Omega_x = \frac{\left(\frac{\partial\Omega}{\partial x}\right)_{t=\bar{t}_f}}{(-\dot{\Omega})}$$

$\bar{\Omega}$ and Ω_x are the parameters to be optimized; it is evident there is no loss of generality in assuming $\dot{\Omega} = -1$.

The system controls for each member of the ensemble are assumed to satisfy

$$u(t) = \sum_{i=0}^{N_u} a_i t^i + \sum_{j=0}^{N_g} b_j t^j \left[x(t) - \sum_{k=0}^{N_x} c_k t^k \right] \tag{17}$$

where N_u, N_g and N_x are specified, a_i, b_j, c_k are unspecified. The first term in (17) is the polynomial approximation to $\bar{u}(t)$. The second term is the result of an assumption that the feedback control is linear in $x(t)$. The $\Sigma c_k t^k$ is the polynomial approximation to $\bar{x}(t)$. $\Sigma b_j t^j$ is the assumed form of the feedback gain. The most general linear feedback would use an $m \times n$ matrix, say $\Lambda(t)$, of unspecified functions of time. Thus, the formulation used here replaces the most general linear feedback control system, which would require storage of $\bar{u}(t)$, $\bar{x}(t)$ and $\Lambda(t)$, by a linear feedback control utilizing polynomial approximations. It may be verified by inspection that a_i, b_j, c_k are $m \times 1$, $m \times n$, $n \times 1$ matrices for each i, j, k respectively.

The problem, then, is to simultaneously choose all parameters $\bar{\Omega}$, Ω_x, a, b, c to minimize J while satisfying the ψ^j mean and variance constraints.

Derivation of Necessary Conditions for the Optimal Parameters

The approach used here will be to adjoin all relevant constraints to the performance index by means of Lagrange multipliers. Hence,

$$J = \mathcal{E}\{\varphi[x(t_f), t_f]\} + \nu^T \mathcal{E}\{\psi[x(t_f), t_f]\} + \frac{1}{2}\sum_j k_j\Big[\big(\psi^j[x(t_f), t_f]\big)^2 - N^j\Big]$$

$$+ \ \mathcal{E}\int_{t_o}^{\bar{t}_f} (\lambda^T + \delta\lambda^T)(f - \dot{x})\, dt \qquad (18)$$

The essential approximation of the analysis is the assumption of "small" perturbations. The ensemble of system trajectories is treated by expanding about the nominal path and keeping terms through quadratic in δx and δu, but dropping higher order terms. As an example:*

$$\mathcal{E}\{\varphi[x(t_f), t_f]\} = \varphi[\bar{x}(\bar{t}_f), \bar{t}_f] + \Big[\frac{\partial\varphi}{\partial x}\mathcal{E}(dx) + \frac{\partial\varphi}{\partial t}\mathcal{E}(dt)\Big]_{t=\bar{t}_f}$$

$$+ \ \frac{1}{2}\mathcal{E}\Big[dx^T \frac{\partial^2\varphi}{\partial x^2} dx + dx^T \frac{\partial^2\varphi}{\partial x\, \partial t} dt + dt \frac{\partial^2\varphi}{\partial t\, \partial x} dx + dt \frac{\partial^2\varphi}{\partial t^2} dt\Big]_{t=\bar{t}_f} \qquad (19)$$

Evaluation of (19) requires evaluation of

$$d[x(t_f)] = x(t_f) - \bar{x}(\bar{t}_f)$$

$$= \delta x(\bar{t}_f) + \int_{\bar{t}_f}^{t_f} \dot{x}(\tau)\, d\tau \qquad (20)$$

But

---

* The ij^{th} element of $\partial^2 h/\partial y\, \partial z$, where h is scalar and y and z are vectors, is defined to be $\dfrac{\partial^2 h}{\partial y^i\, \partial z^j}$.

$$\dot{x}(\tau) = \dot{\bar{x}}(\tau) + \delta\dot{x}(\tau)$$
$$= \dot{\bar{x}}(\bar{t}_f) + \ddot{\bar{x}}(\bar{t}_f)(\tau - \bar{t}_f) + \cdots$$
$$+ \left[\frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial u}\delta u\right]_{t=\bar{t}_f} + \cdots \tag{21}$$

Substituting (21) into (20) and dropping terms above second order gives

$$d[x(t_f)] = \delta x(\bar{t}_f) + \dot{\bar{x}}(\bar{t}_f)\,dt_f + \frac{1}{2}\ddot{\bar{x}}(\bar{t}_f)\,dt_f^{\,2}$$
$$+ \left[\frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial u}\delta u\right]_{t=\bar{t}_f} dt_f \tag{22}$$

Everywhere in (19) that dt_f appears it is replaced by $\bar{\Omega} + \Omega_x\,\delta x(\bar{t}_f)$.* This makes all terminal functions depend only on quantities evaluated at $t = \bar{t}_f$.

The $\dot{x}$ terms in (18) are integrated by parts. The Lagrange multipliers ν are written

$$\nu = \bar{\nu} + d\nu \tag{23}$$

where $d\nu$ is assumed to be of order $\delta x(\bar{t}_f)$. It is further assumed that the Lagrange multiplier functions $\delta\lambda(t)$ are of order $\delta x(t)$.** The Lagrange multipliers k_j are assumed to be order one. These assumptions all rely on the basic assumption that the entire ensemble of trajectories lies within an adequately small neighborhood of the reference path.

Expansion of (18) through second order and grouping similar terms gives

---

* $\bar{\Omega}$ is assumed to be the order of $\mathcal{E}[\delta x(\bar{t}_f)]$.

** Note that $\delta\lambda(t)$ is different on each member of the ensemble, just as $\delta x(t)$ is. $\lambda(t)$ is the same for each member, given by (6) and (10).

$$J = \bar{\Phi} + \left[\frac{\partial\Phi}{\partial x}(I + \dot{x}\,\Omega_x) + \frac{\partial\Phi}{\partial t}\,\Omega_x\right]_{t=\bar{t}_f} \mathcal{E}[\delta x(\bar{t}_f)] + \left[\frac{\partial\Phi}{\partial x}\,\dot{x} + \frac{\partial\Phi}{\partial t}\right]_{t=\bar{t}_f} \bar{\Omega}$$

$$+ \mathcal{E}\left[\frac{\partial\Phi}{\partial x}\,\frac{\partial f}{\partial u}\,\delta u(\bar{\Omega} + \Omega_x\,\delta x)\right]_{t=\bar{t}_f} + \frac{1}{2}\mathcal{E}\left\{\delta x^T\left[\frac{\partial^2\Phi}{\partial x^2} + 2\,\Omega_x^{\;T}\,\frac{\partial\Phi}{\partial x}\,\frac{\partial f}{\partial x}\right.\right.$$

$$+ \left(\frac{\partial\dot{\Phi}}{\partial x}\right)^T\Omega_x + \Omega_x^{\;T}\left(\frac{\partial\dot{\Phi}}{\partial x}\right) + \Omega_x^{\;T}\,\ddot{\Phi}\,\Omega_x + \sum_{j=1}^{p} k_j\left(\frac{\partial\psi^j}{\partial x} + \dot{\psi}^j\,\Omega_x\right)^T\left(\frac{\partial\psi^j}{\partial x} + \dot{\psi}^j\,\Omega_x\right)\Big]$$

$$\left.\delta x\right\}_{t=\bar{t}_f} + \bar{\Omega}\left\{\ddot{\Phi}\,\Omega_x + \frac{\partial\Phi}{\partial x}\,\frac{\partial f}{\partial x} + \left(\frac{\partial\dot{\Phi}}{\partial x}\right) + \sum_{j=1}^{p} k_j\,\dot{\psi}^j\left(\frac{\partial\psi^j}{\partial x} + \dot{\psi}^j\,\Omega_x\right)\right\}_{t=\bar{t}_f} \cdot$$

$$\mathcal{E}[\delta x(\bar{t}_f)] - \sum_{j=1}^{p} k_j\,N^j + \frac{1}{2}\bar{\Omega}^2\left[\ddot{\Phi} + \sum_{j=1}^{p} k_j\,(\dot{\psi}^j)^2\right]_{t=\bar{t}_f} + d\nu^T\Big[\left(\frac{\partial\psi}{\partial x} + \dot{\psi}\,\Omega_x\right)$$

$$\mathcal{E}(\delta x) + \dot{\psi}\,\bar{\Omega}\Big]_{t=\bar{t}_f} - [\lambda^T\mathcal{E}(\delta x)]_{t=\bar{t}_f} + [\lambda^T\mathcal{E}(\delta x)]_{t=t_o} - \mathcal{E}[\delta\lambda^T\,\delta x]_{t=\bar{t}_f}$$

$$+ \mathcal{E}[\delta\lambda^T\,\delta x]_{t=t_o} + \mathcal{E}\int_{t_o}^{\bar{t}_f}\{(\dot{\lambda}^T + \delta\dot{\lambda}^T)(\bar{x} + \delta x) + (\lambda^T + \delta\lambda^T)\bar{f} + \lambda^T\left(\frac{\partial f}{\partial x}\,\delta x\right.$$

$$\left.+ \frac{\partial f}{\partial u}\,\delta u\right) + \frac{1}{2}\left[\delta x^T\,\frac{\partial^2 H}{\partial x^2}\,\delta x + \delta x^T\,\frac{\partial^2 H}{\partial x\,\partial u}\,\delta u + \delta u^T\,\frac{\partial^2 H}{\partial u\,\partial x}\,\delta x\right.$$

$$\left.+ \delta u^T\,\frac{\partial^2 H}{\partial u^2}\,\delta u\right] + \delta\lambda^T\left(\frac{\partial f}{\partial x}\,\delta x + \frac{\partial f}{\partial u}\,\delta u\right)\}\,dt \tag{24}$$

where extensive use has been made of the following notational substitutions:

$$\Phi = \varphi + \bar{\nu}^T\,\psi$$

$$H = \lambda^T f$$

I is the identity matrix

$$(\dot{\ }) = \frac{\partial(\ )}{\partial x}\dot{x} + \frac{\partial(\ )}{\partial t}$$

$$(\ddot{\ }) = \dot{x}^T \frac{\partial^2(\ )}{\partial x^2}\dot{x} + \dot{x}^T \frac{\partial^2(\ )}{\partial x\,\partial t} + \frac{\partial(\ )}{\partial x}\ddot{x} + \frac{\partial^2(\ )}{\partial t\,\partial x}\dot{x} + \frac{\partial^2(\ )}{\partial t^2}$$

and all derivatives are evaluated on the reference path.

Using (12), (13), (17), $\delta u(t)$ may be written as

$$\begin{aligned}\delta u(t) &= u(t) - \bar{u}(t)\\ &= \sum_{i=0}^{N_u} a_i t^i - \bar{u}(t) + \sum_{j=0}^{N_g} b_j t^j\left[\delta x(t) + \bar{x}(t) - \sum_{k=0}^{N_x} c_k t^k\right]\end{aligned} \tag{25}$$

The following purely symbolic notations are introduced for convenience:

$$\sum_{i=0}^{N_u} a_i t^i = at$$

$$\sum_{j=0}^{N_g} b_j t^j = bt$$

$$\sum_{k=0}^{N_x} c_k t^k = ct$$

With these substitutions $\delta u(t)$ may be written as

$$\delta u(t) = at - \bar{u}(t) + bt[\delta x(t) + \bar{x}(t) - ct] \tag{26}$$

$\delta u(t)$ from (26) may be substituted into (24), giving J as a function of $\bar{\Omega}$, Ω_x, a, b, c and other quantities. A necessary condition for optimal choice of the unspecified parameters is that dJ be zero for arbitrary first order changes in the parameters.

By virtue of the optimality of the reference trajectory, all first order terms in dJ, and the $\delta u(\bar{t}_f)$ term also, drop out. Thus, dJ is composed entirely of second order terms and, by a straightforward development, may be written as

$$
\begin{aligned}
dJ = \varepsilon \Bigg[\Bigg\{ \delta x^T \Bigg[& \frac{\partial^2 \Phi}{\partial x^2} + 2\Omega_x^T \frac{\partial \Phi}{\partial x}\frac{\partial f}{\partial x} + \left(\frac{\partial \dot{\Phi}}{\partial x}\right)^T \Omega_x + \Omega_x^T\left(\frac{\partial \dot{\Phi}}{\partial x}\right) + \Omega_x^T \ddot{\Phi}\,\Omega_x \\
& + \sum_{j=1}^{p} k_j \left(\frac{\partial \psi^j}{\partial x} + \dot{\psi}^j \Omega_x\right)^T \left(\frac{\partial \psi^j}{\partial x} + \dot{\psi}^j \Omega_x\right)\Bigg] + \bar{\Omega}\Bigg[\ddot{\Phi}\,\Omega_x + \frac{\partial \Phi}{\partial x}\frac{\partial f}{\partial x} + \left(\frac{\partial \dot{\Phi}}{\partial x}\right) \\
& + \sum_{j=1}^{p} k_j\, \dot{\psi}^j \left(\frac{\partial \psi^j}{\partial x} + \dot{\psi}^j \Omega_x\right)\Bigg] + d\nu^T\left[\frac{\partial \psi}{\partial x} + \dot{\psi}\,\Omega_x\right] - \delta\lambda^T \Bigg\} \delta(\delta x) \Bigg|_{t=\bar{t}_f} \\
& + \Bigg\{ \Bigg[\ddot{\Phi}\,\Omega_x + \frac{\partial \Phi}{\partial x}\frac{\partial f}{\partial x} + \left(\frac{\partial \dot{\Phi}}{\partial x}\right) + \sum_{j=1}^{p} k_j\, \dot{\psi}^j \left(\frac{\partial \psi^j}{\partial x} + \dot{\psi}^j \Omega_x\right)\Bigg] \varepsilon(\delta x) + \bar{\Omega}\Bigg[\ddot{\Phi} \\
& + \sum_{j=1}^{p} k_j (\dot{\psi}^j)^2 \Bigg] + d\nu^T \dot{\psi} \Bigg\}_{t=\bar{t}_f} d\bar{\Omega} + \mathrm{tr}\Bigg\{ X \Bigg[\left(\frac{\partial \dot{\Phi}}{\partial x}\right)^T + \left(\frac{\partial f}{\partial x}\right)^T \left(\frac{\partial \Phi}{\partial x}\right)^T + \Omega_x^T \ddot{\Phi} \\
& + \sum_{j=1}^{p} k_j \left(\frac{\partial \psi^j}{\partial x} + \dot{\psi}^j \Omega_x\right)^T \dot{\psi}^j \Bigg] + \varepsilon(\delta x^T)\Bigg[\overline{\Omega}\ddot{\Phi} + \bar{\Omega}\sum_{j=1}^{p} k_j (\dot{\psi}^j)^2 + d\nu^T \dot{\psi}\Bigg]\Bigg\}_{t=\bar{t}_f} d\Omega_x \\
& + \varepsilon \int_{t_o}^{\bar{t}_f} \Bigg\{ \Bigg[\delta\dot{\lambda}^T + \delta\lambda^T\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u} bt\right) + \delta x^T \Bigg(\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial x\,\partial u} bt + (bt)^T \frac{\partial^2 H}{\partial u\,\partial x} \\
& + (bt)^T \frac{\partial^2 H}{\partial u^2} bt \Bigg) [at - \bar{u} + bt(\bar{x} - ct)]^T \left(\frac{\partial^2 H}{\partial u\,\partial x} + \frac{\partial^2 H}{\partial u^2} bt\right)\Bigg] \delta(\delta x) \\
& + \sum_{i=0}^{N_u} \left[\delta x^T \frac{\partial^2 H}{\partial x\,\partial u} + [at - \bar{u} + bt(\bar{x} - ct)]^T \frac{\partial^2 H}{\partial u^2} + (\delta x^T L + \ell^T)\frac{\partial f}{\partial u}\right] t^i\, da_i
\end{aligned}
$$

$$+\sum_{j=0}^{N_g} \mathrm{tr}\Big[(\delta x+\bar{x}-ct)\Big(\delta x^T\frac{\partial^2 H}{\partial x\,\partial u}+[at-\bar{u}+bt(\delta x+\bar{x}-ct)]^T\frac{\partial^2 H}{\partial u^2}\Big)$$

$$+(\bar{x}-ct)(\delta x^T L+\ell^T)\frac{\partial f}{\partial u}\Big]t^j\, db_j-\sum_{k=0}^{N_x}\Big[\delta x^T\frac{\partial^2 H}{\partial x\,\partial u}+[at-\bar{u}+bt(\delta x+\bar{x}-ct)]^T$$

$$\frac{\partial^2 H}{\partial u^2}+(\delta x^T L+\ell^T)\frac{\partial f}{\partial u}\Big]t^k\, dc_k\Big\}dt \tag{27}$$

where, by definition, tr stands for trace and

$$X(t) = \mathcal{E}[\delta x(t)\,\delta x^T(t)]$$

Setting $dJ = 0$ provides necessary conditions for extremizing choices of the control parameters. The Lagrange multipliers $\delta\lambda$ satisfy

$$\delta\dot{\lambda}+\Big(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial u}bt\Big)^T\delta\lambda+\Big(\frac{\partial^2 H}{\partial x^2}+(bt)^T\frac{\partial^2 H}{\partial u\,\partial x}+\frac{\partial^2 H}{\partial x\,\partial u}bt+(bt)^T\frac{\partial^2 H}{\partial u^2}bt\Big)\delta x$$

$$+\Big(\frac{\partial^2 H}{\partial x\,\partial u}+(bt)^T\frac{\partial^2 H}{\partial u^2}\Big)[at-\bar{u}+bt(\bar{x}-ct)] = 0 \tag{28}$$

$$\delta\lambda(\bar{t}_f) = \Big\{\frac{\partial^2\Phi}{\partial x^2}+2\Big(\frac{\partial f}{\partial x}\Big)^T\Big(\frac{\partial\Phi}{\partial x}\Big)^T\Omega_x+\Big(\frac{\partial\dot{\Phi}}{\partial x}\Big)^T\Omega_x+\Omega_x^T\Big(\frac{\partial\dot{\Phi}}{\partial x}\Big)+\Omega_x^T\ddot{\Phi}\,\Omega_x$$

$$+\sum_{j=1}^{p}k_j\Big(\frac{\partial\psi^j}{\partial x}+\dot{\psi}^j\Omega_x\Big)^T\Big(\frac{\partial\psi^j}{\partial x}+\dot{\psi}^j\Omega_x\Big)+\bar{\Omega}\Big[\ddot{\Phi}\,\Omega_x+\frac{\partial\Phi}{\partial x}\frac{\partial f}{\partial x}+\Big(\frac{\partial\dot{\Phi}}{\partial x}\Big)$$

$$+\sum_{j=1}^{p}k_j\,\dot{\psi}^j\Big(\frac{\partial\psi^j}{\partial x}+\dot{\psi}^j\Omega_x\Big)\Big]+d\nu^T\Big(\frac{\partial\psi}{\partial x}+\dot{\psi}\,\Omega_x\Big)\Big\}_{t=\bar{t}_f} \tag{29}$$

Because $\delta x(t)$ is a vector of random variables, $\delta\lambda(t)$ is also. Neither can be used computationally. However, it may be verified by direct substitution that

$$\delta\lambda(t) = L(t)\,\delta x(t) + \ell(t) \tag{30}$$

where $L(t)$ and $\ell(t)$ are the same for every member of the ensemble.

$$\dot{L} + \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}\,bt\right)^T L + L\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}\,bt\right) + \frac{\partial^2 H}{\partial x^2} + (bt)^T \frac{\partial^2 H}{\partial u\,\partial x}$$

$$+ \frac{\partial^2 H}{\partial x\,\partial u}\,bt + (bt)^T \frac{\partial^2 H}{\partial u^2}\,bt = 0 \tag{31}$$

$$\dot{\ell} + \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}\,bt\right)^T \ell + \left(\frac{\partial^2 H}{\partial x\,\partial u} + (bt)^T \frac{\partial^2 H}{\partial u^2}\right)\left(at - \overline{u} + bt(\overline{x} - ct)\right) = 0 \tag{32}$$

The boundary conditions for L and ℓ are evident by inspection of (29).

To obtain the remainder of the necessary conditions resulting from $dJ = 0$, it is necessary to develop the differential equations for $\mathcal{E}(\delta x)$ and for X. First, it may be noted that $\mathcal{E}(\delta x)$ appears only in terms that are second order, hence it need be calculated only to first order. The linearized perturbations of (1) with $\delta u(t)$ from (26) immediately give

$$\frac{d}{dt}\,\mathcal{E}(\delta x) = \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}\,bt\right)\mathcal{E}(\delta x) + \frac{\partial f}{\partial u}\left[at - \overline{u} + bt(\overline{x} - ct)\right] \tag{33}$$

It is convenient to define

$$\delta x = \mathcal{E}(\delta x) + \delta\tilde{x} \tag{34}$$

so that

$$X = \mathcal{E}(\delta x)\,\mathcal{E}(\delta x^T) + \tilde{X} \tag{35}$$

$$\tilde{X} = \mathcal{E}[\delta\tilde{x}\;\delta\tilde{x}^T] \tag{36}$$

Then, by direct substitution

$$\dot{\tilde{X}} = \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}\,bt\right)\tilde{X} + \tilde{X}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}\,bt\right)^T \tag{37}$$

The boundary conditions for $\mathcal{E}(\delta x)$ and $\tilde{X}$ are given by

$$\mathcal{E}[\delta x(t_o)]\,,\quad \text{specified} \tag{38}$$

$$\mathcal{E}[\delta x(t_o)\;\delta x^T(t_o)] = X(t_o),\quad \text{specified} \tag{39}$$

There are thus $2(n^2 + n)$ differential equations for $\mathcal{E}(\delta x)$, $\tilde{X}$, ℓ, L and corresponding boundary conditions, half at t_o and half at $\bar{t}_f$. The conditions at $\bar{t}_f$ involve the Lagrange multipliers $d\nu$ and k_j; constraint equations (3) and (4) furnish the additional required $2p$ relations.

From (16) it is clear that $\bar{\Omega}$ is a bias in the choice of dt_f. Such a bias gives added flexibility because the differences of at and ct from $\bar{u}$ and $\bar{x}$ respectively cause $\mathcal{E}[\delta x(t)]$ to be non-zero. Applying $dJ = 0$, $\bar{\Omega}$ may be explicitly solved for in terms of other parameters of the problem:

$$\bar{\Omega} = -\left\{\frac{\left[\ddot{\Phi}\,\Omega_x + \frac{\partial \Phi}{\partial x}\frac{\partial f}{\partial x} + \left(\frac{\partial \dot{\Phi}}{\partial x}\right) + \sum_{j=1}^{p} k_j\,\dot{\psi}^j\left(\frac{\partial \psi^j}{\partial x} + \dot{\psi}^j\,\Omega_x\right)\right]\mathcal{E}(\delta x) + d\nu^T\,\dot{\psi}}{\ddot{\Phi} + \sum_{j=1}^{p} k_j\,(\dot{\psi}^j)^2}\right\}_{t=\bar{t}_f} \tag{40}$$

Thus, $\bar{\Omega}$ need not appear as an unknown in any numerical optimization procedure.

The parameters Ω_x may also be solved for, from $dJ = 0$, in terms of quantities evaluated at $t = \bar{t}_f$:

$$\Omega_x = -\left\{ \frac{\left[\bar{\Omega}\ddot{\Phi} + \bar{\Omega}\sum_{j=1}^{p} k_j(\dot{\psi}^j)^2 + d\nu^T\dot{\psi}\right]\mathcal{E}(\delta x)X + \left(\frac{\partial\dot{\Phi}}{\partial x}\right) + \frac{\partial f}{\partial x}\frac{\partial \Phi}{\partial x} + \sum_{j=1}^{p} k_j\dot{\psi}^j\frac{\partial\dot{\psi}^j}{\partial x}}{\ddot{\Phi} + \sum_{j=1}^{p} k_j(\dot{\psi}^j)^2} \right\} \quad (41)$$

After utilizing (40) and (41), the unspecified parameters are a, b, c. These must satisfy integral relations which result from $dJ = 0$, for arbitrary small changes da, db, dc.

$$0 = \int_{t_o}^{\bar{t}_f} \left\{ \mathcal{E}(\delta x^T)\frac{\partial^2 H}{\partial x \partial u} + [at - \bar{u} + bt(\bar{x} - ct)]^T \frac{\partial^2 H}{\partial u^2} + [\mathcal{E}(\delta x^T)L + \ell^T]\frac{\partial f}{\partial u}\right\} t^i \, dt \quad (42)$$

$$i = 0, 1, 2, --, N_u$$

$$\begin{aligned} 0 = \int_{t_o}^{\bar{t}_f} \Big\{ & [X + (\bar{x} - ct)\mathcal{E}(\delta x^T)]\frac{\partial^2 H}{\partial x \partial u} + \{X(bt)^T + \mathcal{E}(\delta x)[at - \bar{u} + bt(\bar{x} - ct)]^T \frac{\partial^2 H}{\partial u^2} \\ & + (\bar{x} - ct)\mathcal{E}(\delta x^T)(bt)^T + (\bar{x} - ct)[at - \bar{u} + bt(\bar{x} - ct)]^T\}\frac{\partial^2 H}{\partial u^2} \\ & + (\bar{x} - ct)[\mathcal{E}(\delta x^T)L + \ell^T]\frac{\partial f}{\partial u}\Big\} t^j \, dt \end{aligned} \quad (43)$$

$$j = 0, 1, 2, --, N_g$$

$$0 = \int_{t_o}^{\bar{t}_f} \left\{ \mathcal{E}(\delta x^T)\frac{\partial^2 H}{\partial x\, \partial u} + [at - \bar{u} + bt(\mathcal{E}(\delta x) + \bar{x} - ct)]^T \frac{\partial^2 H}{\partial u^2} + [\mathcal{E}(\delta x^T)L + \ell^T]\frac{\partial f}{\partial u}\right\} t^k dt \tag{44}$$

$$k = 0, 1, 2, --, N_x$$

The parameters a, b, c may not be eliminated algebraically because other quantities depend on them. The necessary conditions involving $\mathcal{E}(\delta x)$, $\widetilde{X}$, ℓ, L, a, b, c are all interlocked. This is characteristic of dynamic system optimization problems with control parameters. Although such problems are seldom easy, the one considered here presents no new conceptual difficulties.

An Alternative Approach to the Necessary Conditions Derivation

This analysis is based on second order expansions and is closely related to the second variation guidance schemes of Refs. 1 and 2. There, for a single perturbed trajectory, the second variation of the performance index is minimized subject to satisfaction of the $\dot{x} = f$ and $\psi = 0$ constraints. One proceeds by making stationary the function $\Phi = \varphi + \bar{\nu}^T \psi$, where properly chosen $\bar{\nu}$ will lead to satisfaction of the terminal constraints. The second variation of Φ, from Refs. 1 and 2, is

$$J_2 = \left[dx^T \frac{\partial^2 \Phi}{\partial x^2} dx + dx^T \frac{\partial^2 \Phi}{\partial x \, \partial t} dt + dt \frac{\partial^2 \Phi}{\partial t \, \partial x} dx + dt \frac{\partial^2 \Phi}{\partial t^2} dt \right]_{t = \bar{t}_f}$$

$$+ \int_{t_o}^{\bar{t}_f} \left[\delta x^T \frac{\partial^2 H}{\partial x^2} \delta x + \delta x^T \frac{\partial^2 H}{\partial x \partial u} \delta u + \delta u^T \frac{\partial^2 H}{\partial u \partial x} \delta x + \delta u^T \frac{\partial^2 H}{\partial u^2} \delta u \right] dt \quad (45)$$

Since the reference path satisfies all the constraints, it is sufficient to adjoin the linearized perturbation constraints

$$\delta \dot{x} = \frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial u} \delta u \quad (46)$$

$$d\psi = \left[\frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial t} dt \right]_{t = \bar{t}_f} = 0 \quad (47)$$

Then, given $\delta x(t_o)$, $\delta u(t)$ is chosen to minimize $\frac{1}{2} J_2$ while satisfying constraints (46) and (47). This leads to a linear feedback relation

$$\delta u(t) = - \Lambda(t) \, \delta x(t) \quad (48)$$

It is tacitly assumed that $\bar{x}(t)$ and $\bar{u}(t)$ as well as $\Lambda(t)$ are "stored" (available to the guidance system).

The significant operational simplification of neighboring extremal guidance introduced in this paper is the substitution of a relatively small number of polynomial coefficients for the functions $\bar{u}(t)$, $\bar{x}(t)$, $\Lambda(t)$. The general functions of time would require tables of values vs. time in operation with a digital computer. Use of polynomial coefficients instead may be expected to greatly reduce the storage requirements.

An additional advantage of the polynomial approximations is that the difficulty $\Lambda(t) \to \infty$ as $t \to \bar{t}_f$ disappears. The polynomial $\Sigma b_j t^j$ will certainly be well behaved in the neighborhood of $t = \bar{t}_f$. Thus, the need for a transverse state comparison, so important for neighboring extremal control, may become less significant in analyses conducted along the present lines.

It is, of course, necessary to satisfy the constraint (46) in any (small perturbation) analysis. It is not possible, however, to satisfy (47) for arbitrary $\delta x(t_o)$ with the polynomial approximations. Hence, the use of a statistical performance index is not only appropriate, but even unavoidable. The alternative approach to the derivation of the previous section is to consider minimizing the ensemble average of $\frac{1}{2} J_2$. Constraints on the mean and variance of the ψ^j's [equations (3) and (4)] are imposed. Because these ensemble averages involve only the mean and covariance of $\delta x(t)$, it is sufficient to use the differential equations for $\mathcal{E}(\delta x)$ and X in place of (46). Thus, (24) is fully equivalent to

$$J = \mathcal{E}[\tfrac{1}{2} J_2] + d\nu^T \mathcal{E}\{d\psi[x(t_f), t_f]\} + \sum_{j=1}^{p} \frac{1}{2} k_j \, \mathcal{E}\{d\psi^j[x(t_f), t_f]\}^2 + \int_{t_o}^{\bar{t}_f} \Big\{ \ell^T \Big[\Big(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u} bt\Big)\mathcal{E}(\delta x)$$

$$+ [at - \bar{u} + bt(\bar{x} - ct)] - \frac{d}{dt}\mathcal{E}(\delta x)\Big] + \mathrm{tr}\, L\Big[\Big(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u} bt\Big)\tilde{X} + \tilde{X}\Big(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u} bt\Big)^T$$

$$+ \frac{d}{dt}[\mathcal{E}(\delta x)\,\mathcal{E}(\delta x^T)] - \dot{X}\Big]\Big\} dt \qquad (49)$$

Here $\ell(t)$ and $L(t)$ appear as a vector and matrix respectively of Lagrange multiplier functions.* $\ell(t)$ is the vector adjoint to $\delta\mathcal{C}[\delta x(t)]$, $L(t)$ is the matrix adjoint to $\delta X(t)$. All the necessary conditions of the previous section may be obtained by requiring J of (49) to be stationary with respect to arbitrary small changes in the unspecified parameters.

---

* Since L multiplies symmetric matrices in (49), it may be assumed symmetric with no loss of generality.

Possible Additional Complexities

The analysis as presented allows disturbances only in the form of perturbations in the initial state variables. It also assumes that the state is known perfectly at all times. Both restrictions may be relaxed while still retaining the polynomial approximation approach.

Disturbing influences may arise from perturbations of system parameters from their reference values. For example, the thrust and/or fuel consumption rate of a rocket vehicle may deviate from its pre-planned value. To allow for this in the analysis presented here, such system parameters may be regarded as state variables with zero time derivatives. Thus, a parameter deviation becomes an initial state variable perturbation.

Time-dependent random forcing functions may be added to the analysis if their means and covariances are known, although serious complications may arise if the noise is appreciably correlated in time. The main effect with zero-mean white noise would be to add a term to $\dot{X}$. The other equations would be unaltered, but any numerical solution might be substantially different.

If state estimation errors were not considered negligible, it would be possible to include them by considering the estimator characteristics. A linear perturbation estimator would be consistent with the degree of approximation used here. The estimator gain matrix would play a role analogous to the feedback gain matrix. It would be approximated by a polynomial analogous to $\Sigma b_j t^j$. The polynomial coefficients would be added to the others, all to be chosen simultaneously to optimize the system ensemble average performance.

Computational Considerations

The preceding analysis has been devoted to problem formulation and development of first order necessary conditions for a minimum. Computational determination of the control parameters which actually furnish a minimum represents a second phase of study. It is clear, however, that any of the methods applicable to the solution of Mayer/Bolza variational problems appear likely to be equally suitable to parameter optimization problems of the present type. On the basis of experience, the writers are favorably inclined toward the use of gradient methods (Refs. 7 and 8) and methods of the second variation type (Ref. 9), and in this connection it should be noted that the usual requirement for rapid access storage of control variables versus time is eased in favor of a somewhat less severe requirement for storage of parameter values. With the second order method of Ref. 9, it appears that parameter optimization will entail the solution of fairly large linear algebraic systems, and hence that greater attention than usual must be given to error propagation problems.

Concluding Remarks

The present paper has sketched in some detail an ensemble averaging approach to optimal guidance polynomial approximations. Conclusions on the merits of this approach must be deferred until numerical examples of synthesis procedure have been worked and system simulations performed. In connection with the problem of guidance system mechanization, it will be of interest to investigate the use of transverse state comparison or some similar mode of comparison employing polynomial representation.

References

1. Kelley, H.J.; "Guidance Theory and Extremal Fields," IRE National Aerospace Electronics Conference, May 14-16, 1962; also IRE Transactions on Automatic Control, October 1962.

2. Breakwell, J.V. and Bryson, A.E.; "Neighboring-Optimum Terminal Control for Multivariable Nonlinear Systems," SIAM Symposium on Multivariable System Theory, Cambridge, Mass., Nov. 1-3, 1962; also
Breakwell, J.V., Speyer, J.L. and Bryson, A.E.; "Optimization and Control of Nonlinear Systems Using the Second Variation," JSIAM Control, Vol. 1, No. 2, 1963.

3. Kelley, H.J.; "An Optimal Guidance Approximation Theory," IEEE Transactions on Automatic Control, October 1964.

4. Dunn, J.C.; "Autonomous Time-Optimal Extremal Fields," to appear.

5. Striebel, C.T. and Breakwell, J.V.; "Minimum Effort Control in Interplanetary Guidance," IAS 31st Annual Meeting, New York, N.Y., January 21-23, 1963.

6. Denham, W.F. and Speyer, J.L.; "Optimal Measurement and Velocity Correction Programs for Midcourse Guidance," AIAA Summer Meeting, Los Angeles, Calif., June 17-20, 1963; also AIAA Journal, Vol. 2, No. 5, May 1964.

7. Kelley, H.J.; "Method of Gradients," Chapter 6 of Optimization Techniques, G. Leitmann, Editor, Academic Press, New York, 1962.

8. Bryson, A.E. and Denham, W.F.; "A Steepest-Ascent Method for Solving Optimum Programming Problems," Journal of Applied Mechanics, June 1962.

9. Kelley, H.J., Kopp, R.E. and Moyer, H.G.; "A Trajectory Optimization Technique Based Upon the Theory of the Second Variation," AIAA Astrodynamics Conference, New Haven, Conn., August 19-21, 1963. Also in Progress in Astronautics and Aeronautics, Vol. 14, Academic Press, New York, 1964.